FORM 3TBP

1997

The College Board
Advanced Placement Examination

CALCULUS BC
SECTION II

This green insert may be used for reference and/or scratchwork as you answer the free-response questions, but be sure to show all your work and your answers in the pink booklet. No credit will be given for work shown on this green insert.

1. During the time period from $t = 0$ to $t = 6$ seconds, a particle moves along the path given by $x(t) = 3\cos(\pi t)$ and $y(t) = 5\sin(\pi t)$.

   (a) Find the position of the particle when $t = 2.5$.

   (b) On the axes provided below, sketch the graph of the path of the particle from $t = 0$ to $t = 6$. Indicate the direction of the particle along its path.

   (c) How many times does the particle pass through the point found in part (a)?

   (d) Find the velocity vector for the particle at any time t.

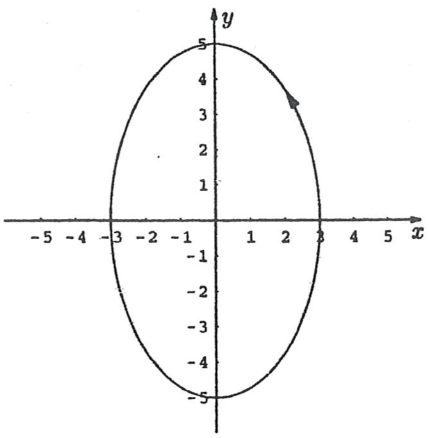
   (e) Write and evaluate an integral expression, in terms of sine and cosine, that gives the distance the particle travels from time $t = 1.25$ to $t = 1.75$.

(a) $x(2.5) = 3\cos(2.5\pi) = 0$

$y(2.5) = 5\sin(2.5\pi) = 5$

1: $x(2.5) = 0$ and $y(2.5) = 5$

(b)



$2 \begin{cases} \text{1: graph: ellipse with vertices } (\pm 3, 0) \text{ and } (0, \pm 5) \\ \text{1: direction: counterclockwise} \end{cases}$

(c) 3

1: answer

(d) $x'(t) = -3\pi\sin(\pi t)$ $\quad y'(t) = 5\pi\cos(\pi t)$

$\vec{v}(t) = \langle -3\pi\sin(\pi t), 5\pi\cos(\pi t) \rangle$

$2 \begin{cases} \text{1: } x'(t) \text{ and } y'(t) \\ \text{1: vector answer} \end{cases}$

(e) distance

$$= \int_{1.25}^{1.75} \sqrt{9\pi^2 \sin^2(\pi t) + 25\pi^2 \cos^2(\pi t)}\, dt$$

$= 5.392$

3:
- 2: arclength integral
  - <−1> each arithmetic error
  - <−1> error in limits
- 1: answer
  (eligible only if at least 1 integral point has been earned)

BC-2

2. Let $P(x) = 7 - 3(x-4) + 5(x-4)^2 - 2(x-4)^3 + 6(x-4)^4$ be the fourth-degree Taylor polynomial for the function f about 4. Assume f has derivatives of all orders for all real numbers.

(a) Find $f(4)$ and $f'''(4)$.

(b) Write the second-degree Taylor polynomial for f' about 4 and use it to approximate $f'(4.3)$.

(c) Write the fourth-degree Taylor polynomial for $g(x) = \int_4^x f(t)\,dt$ about 4.

(d) Can $f(3)$ be determined from the information given? Justify your answer.

(a) $f(4) = P(4) = 7$

$$\frac{f'''(4)}{3!} = -2, \quad f'''(4) = -12$$

2 { 1: $f(4) = 7$; 1: $f'''(4) = -12$ }

(b) $P_3(x) = 7 - 3(x-4) + 5(x-4)^2 - 2(x-4)^3$

$P_3'(x) = -3 + 10(x-4) - 6(x-4)^2$

$f'(4.3) \approx -3 + 10(0.3) - 6(0.3)^2 = -0.54$

3 {
- 2: $P_3'(x)$
  - <−1> for each incorrect term
  - <−1> for each additional term or + . . .
  - 0/2 if degree < 2
- 1: evaluation (only if degree 1, 2, 3)

}

(c) $P_4(g, x) = \int_4^x P_3(t)\,dt$

$$= \int_4^x \left[7 - 3(t-4) + 5(t-4)^2 - 2(t-4)^3\right] dt$$

$$= 7(x-4) - \frac{3}{2}(x-4)^2 + \frac{5}{3}(x-4)^3 - \frac{1}{2}(x-4)^4$$

2 {
- 1: $P_4(g, x) = \int_4^x P_3(t)\,dt$ or $P_4(g, x) = \int_4^x P(t)\,dt$ (ignore + . . .)
- 1: answer

}

(d) No. The information given provides values for $f(4)$, $f'(4)$, $f''(4)$, $f'''(4)$, and $f^{(4)}(4)$ only.

2 { 1: answer; 1: reason }

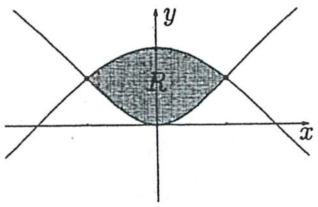
3. Let R be the region enclosed by the graphs of $y = \ln(x^2+1)$ and $y = \cos x$.

(a) Find the area of R.

(b) Write an expression involving one or more integrals that gives the length of the boundary of the region R. Do not evaluate.

(c) The base of a solid is the region R. Each cross section of the solid perpendicular to the x-axis is an equilateral triangle. Write an expression involving one or more integrals that gives the volume of the solid. Do not evaluate.

(a)



$\ln(x^2+1) = \cos x$

$x = \pm 0.91586$

let $B = 0.91586$

$$\text{area} = \int_{-B}^{B} [\cos x - \ln(x^2+1)]\, dx$$

$$= 1.168$$

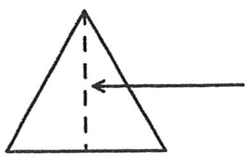
3 {
- 2: integral
  - 1: limits
  - 1: integrand
- 1: answer

(b)

$$L = \int_{-B}^{B} \sqrt{1 + \left(\frac{2x}{x^2+1}\right)^2}\, dx + \int_{-B}^{B} \sqrt{1 + (-\sin x)^2}\, dx$$

4 {
- 2: derivatives of $\cos x$ and $\ln(x^2+1)$
- 1: arclength integrands
- 1: limits and sum of integrals

(c)



$$\text{area of cross section} = \frac{1}{2}\left[\cos x - \ln(x^2+1)\right] \times \left[\frac{\sqrt{3}}{2}\left(\cos x - \ln(x^2+1)\right)\right]$$

$$\text{volume} = \int_{-B}^{B} \frac{\sqrt{3}}{4}\left[\cos x - \ln(x^2+1)\right]^2 dx$$

2 {
- 1: $V = \int_{-0.916}^{0.916} k\left[\cos x - \ln(x^2+1)\right]^2 dx$
- 1: constant k

1/2 if sections perpendicular to y-axis

$$V = \sqrt{3}\int_0^{.609} (e^y - 1)\, dy + \sqrt{3}\int_{.609}^{1} (\cos^{-1} y)^2\, dy$$

4. Let $x = ky^2 + 2$, where $k > 0$.

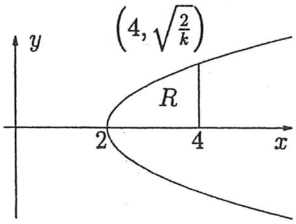
(a) Show that for all $k > 0$, the point $\left(4, \sqrt{\frac{2}{k}}\right)$ is on the graph of $x = ky^2 + 2$.

(b) Show that for all $k > 0$, the tangent line to the graph of $x = ky^2 + 2$ at the point $\left(4, \sqrt{\frac{2}{k}}\right)$ passes through the origin.

(c) Let R be the region in the first quadrant bounded by the x-axis, the graph of $x = ky^2 + 2$, and the line $x = 4$. Write an integral expression for the area of the region R and show that this area decreases as k increases.

(a) $4 = k\left(\frac{2}{k}\right) + 2$

$4 = 4$



1: substitutes $x = 4$ and $y = \sqrt{\frac{2}{k}}$

(b) $x = ky^2 + 2$

$1 = 2ky\,\frac{dy}{dx}$

$\left.\frac{dy}{dx}\right|_{y=\sqrt{2/k}} = \frac{1}{2\sqrt{2k}}$

the tangent line is

$y - \sqrt{\frac{2}{k}} = \frac{1}{2\sqrt{2k}}(x - 4)$

$y = \frac{1}{2\sqrt{2k}}x$ which contains $(0,0)$

or

slope of line through $(0,0)$ and $\left(4, \sqrt{2/k}\right)$

is $\frac{\sqrt{2/k}}{4} = \frac{1}{2\sqrt{2k}}$

which is the same as the slope of the tangent line

4 {
- 1: equation with $\frac{dy}{dx}$ or $\frac{dx}{dy}$
- 1: $\frac{dy}{dx}$ or $\frac{dx}{dy}$ at $\left(4, \sqrt{\frac{2}{k}}\right)$
- 1: equation of tangent line
  or
  slope of line using $(4, \sqrt{2/k})$ and $(0,0)$
- 1: observes $(0,0)$ is a solution
  or
  observes the slopes are equal

(c) $A = \int_0^{\sqrt{2/k}} \left(4 - (ky^2 + 2)\right)\, dy$

or

$A = \frac{1}{\sqrt{k}} \int_2^4 \sqrt{x - 2}\, dx$

$A = \frac{4\sqrt{2}}{3} k^{-0.5}$

$\frac{dA}{dk} = -\frac{2\sqrt{2}}{3} k^{-1.5} < 0$ for all $k > 0$

thus the area decreases as k increases

4 {
- 3: expresses the area as a definite integral

  $A = \int_0^{\sqrt{2/k}} \left(4 - (ky^2 + 2)\right)\, dy$

  or

  $A = \frac{1}{\sqrt{k}} \int_2^4 \sqrt{x - 2}\, dx$

  $<-1>$ error in limits

  $<-1>$ considers twice the region

  $<-2>$ error in integrand
- 1: shows that A decreases using algebra, geometry, or calculus

6. Let $v(t)$ be the velocity, in feet per second, of a skydiver at time t seconds, $t \geq 0$. After her parachute opens, her velocity satisfies the differential equation $\frac{dv}{dt} = -2v - 32$, with initial condition $v(0) = -50$.

(a) Use separation of variables to find an expression for v in terms of t, where t is measured in seconds.

(b) Terminal velocity is defined as $\lim_{t\to\infty} v(t)$. Find the terminal velocity of the skydiver to the nearest foot per second.

(c) It is safe to land when her speed is 20 feet per second. At what time t does she reach this speed?

(a) $\dfrac{dv}{dt} = -2v - 32 = -2(v+16)$

$\dfrac{dv}{v+16} = -2\,dt$

$\ln|v+16| = -2t + A$

$|v+16| = e^{-2t+A} = e^{A}e^{-2t}$

$v + 16 = Ce^{-2t}$

$-50 + 16 = Ce^{0}\ ; \quad C = -34$

$v = -34e^{-2t} - 16$

6:
- 1: separates variables
- 1: antiderivative of dv side
  - 0/1 if not $\displaystyle\int \frac{dv}{av+b}$, $a \neq 0$
- 1: antiderivative of dt side
- 1: constant of integration
- 1: uses initial condition $v(0) = -50$
- 1: solves for $v(t)$
  - 0/1 if not solving $\dfrac{dv}{av+b} = k\,dt$ where a, b, k nonzero
  - 0/1 if no constant of integration

0/6 if variables not separated

(b) $\lim_{t\to\infty} v(t) = \lim_{t\to\infty}\left(-34e^{-2t} - 16\right) = -16$

1: limit value
must be exponential $v(t)$ with finite limit

(c) $v(t) = -34e^{-2t} - 16 = -20$

$e^{-2t} = \dfrac{2}{17}\ ; \quad t = -\dfrac{1}{2}\ln\left(\dfrac{2}{17}\right) = 1.070$

2:
- 1: sets $v(t) = -20$
- 1: solution
  - must be exponential $v(t)$

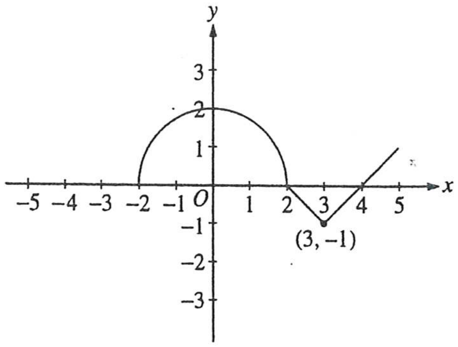
5. The graph of a function f consists of a semicircle and two line segments as shown above. Let g be the function given by $g(x) = \int_0^x f(t)\,dt$.

(a) Find $g(3)$.

(b) Find all values of x on the open interval $(-2, 5)$ at which g has a relative maximum. Justify your answer.

(c) Write an equation for the line tangent to the graph of g at $x = 3$.

(d) Find the x-coordinate of each point of inflection of the graph of g on the open interval $(-2, 5)$. Justify your answer.



(a) $g(3) = \int_0^3 f(t)\,dt$

$$= \frac{1}{4}\cdot\pi\cdot 2^2 - \frac{1}{2} = \pi - \frac{1}{2}$$

2 { 2: answer; <−1> each incorrect area; <−1> error in summing }

(b) $g(x)$ has relative maximum at $x = 2$

because $g'(x) = f(x)$ changes from positive to negative at $x = 2$

3 { 1: relative maximum at $x = 2$ only; 1: $g'(x) = f(x)$ or interprets $g(x)$ as area accumulator; 1: justification (ignore discussion at $x = 5$) }

(c) $g(3) = \pi - \frac{1}{2}$

$g'(3) = f(3) = -1$

$$y - \left(\pi - \frac{1}{2}\right) = -1(x-3)$$

2 { 1: $g'(3) = -1$; 1: equation using $g(3)$ and $g'(3)$ }

(d) graph of g has points of inflection with x-coordinates $x = 0$ and $x = 3$

because $g''(x) = f'(x)$ changes from positive to negative at $x = 0$ and from negative to positive at $x = 3$

or

because $g'(x) = f(x)$ changes from increasing to decreasing at $x = 0$ and from decreasing to increasing at $x = 3$

2 { 1: points of inflection with x-coordinates 0 and 3 only; 1: justification (ignore discussion at $x = 2$) }

1/2 if $x = 0, 3$ selected as candidates and $x = 3$ discarded because $g''(3)$ does not exist

1/2 if $x = 0, 2, 3$ selected as candidates and $x = 2$ and $x = 3$ discarded because $g''(2)$ and $g''(3)$ do not exist

AB-6, BC-6
Board Note # 1

Part (a) Integrating Factor Solution: Max 5/6

$v'(t) = -2v(t) - 32$

$e^{2t}\left(v'(t) + 2v(t)\right) = -32e^{2t}$ — 1: use of integrating factor

$\frac{d}{dt}\left(v(t)e^{2t}\right) = -32e^{2t}$

$v(t)e^{2t} = \int -32e^{2t}\,dt = -16e^{2t} + C$

$v(t) = -16 + Ce^{-2t}$

$-50 = -16 + C \;;\quad C = -34$

$v(t) = -16 - 34e^{-2t}$ — 4: solution